# Module 2: Lecture 8

# Introduction to Differential Equations

## Definition 1.1

**Differential Equations (DE)**
» equations containing derivatives of one or more variables

$$y''' - 3y' + 2y = 0 \quad (1)$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}i = E\omega\cos(\omega t) \quad (2)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (3)$$

$$(x + y)dx + (3x^2 - 1)dy = 0 \quad (4)$$

## Definition 1.2

When an equation involves derivative(s) with respect to a particular variable, that variable is called an independent variable. A variable is called dependent if a derivative of that variable occurs.

In the Differential Equation,

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{c}i=E\omega\cos(\omega t)$$

$t$ is the independent variable and $i$ is the dependent variable.

## Definition 1.3

An ordinary differential equation (ODE) is a differential equation in which all derivatives are ordinary derivatives of one or more dependent variables with respect to a single dependent variable.

$$y''' - 3y' + 2y = 0$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}i = E\omega\cos(\omega t)$$

$$(x + y)dx + (3x^2 - 1)dy = 0$$

## Definition 1.5

A partial differential equation (PDE) is a differential equation containing at least one partial derivative of some dependent variable.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

## Definition 1.7

The order of the highest-ordered derivative that appears in a DE is called the order of the DE.

$y''' - 3y' + 2y = 0$ order 3

$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}i = E\omega\cos(\omega t)$ and

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ order 2

$(x + y)dx + (3x^2 - 1)dy = 0$ order 1

## Definition 2.1

A solution of an ordinary differential equation in one dependent variable or an interval $I$ is a function which, when substituted for the dependent variable, reduces the equation to an identity in the independent variable over $I$.

| Equation | Solution |
|---|---|
| $y'=2x$ | $y=x^2$ |
| $y'+3y=6x+11$ | $y=e^{-3x}+2x+3$ |
| $y''-3y'+2y=24e^{-2x}$ | $y=3e^{x}-4e^{2x}+2e^{-2}$ |

Remark:
Note that the solution to a DE is not necessarily unique, primarily because the derivative of a constant is zero. For example: $y=x^2+4$ is also a solution to the first differential equation $y'=2x$

Example:

Verify that the function $y=e^{-3x}+2x+3$ is a solution to the differential equation $y'+3y=6x+11$

To verify the solution, we first calculate $y'$ using the chain rule for derivatives.

$$y = e^{-3x} + 2x + 3$$

$$y' = -3e^{-3x} + 2.$$

Next we substitute $y$ and $y'$ into the left-hand side of the differential equation: $y' + 3y = 6x + 11$

$$\begin{aligned} y' + 3y &= (-3e^{-3x} + 2) + 3(e^{-3x} + 2x + 3) \\ &= -3e^{-3x} + 2 + 3e^{-3x} + 6x + 9 \\ &= 6x + 11 \end{aligned}$$

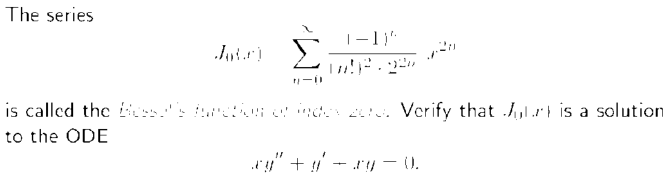
This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

## Activity 2.2

Verify that the given equation is a solution to the corresponding ODE:

- $y = e^{2x}$; $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$
- $y = \sin(kx)$; $\frac{d^2y}{dx^2} + k^2y = 0$
- $y = 3e^{-2x} + 4e^{x}$; $y''' - 3y' + 2y = 0$

## Activity 2.3



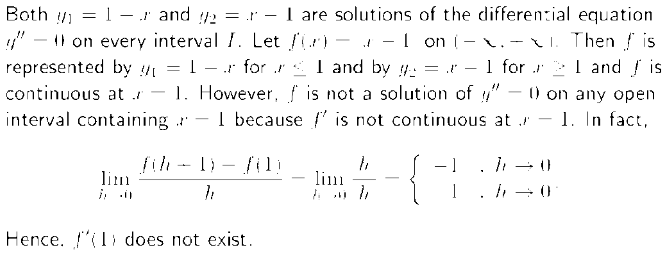
The series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2 \cdot 2^{2n}} x^{2n}$$

is called the *Bessel's function of index zero*. Verify that $J_0(x)$ is a solution to the ODE

$$xy'' + y' + xy = 0.$$

## Example 2.4



Both $y_1 = 1 - x$ and $y_2 = x - 1$ are solutions of the differential equation $y'' = 0$ on every interval $I$. Let $f(x) = |x - 1|$ on $(-\infty, \infty)$. Then $f$ is represented by $y_1 = 1 - x$ for $x \le 1$ and by $y_2 = x - 1$ for $x \ge 1$ and $f$ is continuous at $x = 1$. However, $f$ is not a solution of $y'' = 0$ on any open interval containing $x = 1$ because $f'$ is not continuous at $x = 1$. In fact,

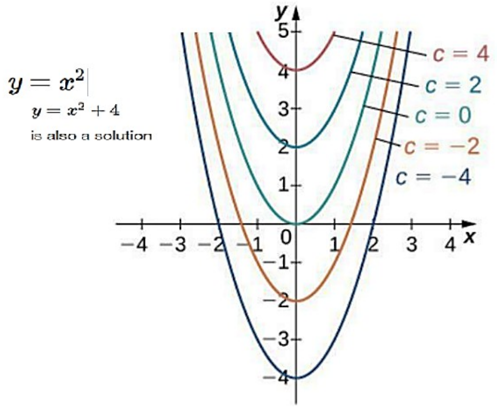
$$\lim_{h \to 0} \frac{f(h+1) - f(1)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} -1 & , h \to 0^- \\ 1 & , h \to 0^+ \end{cases}$$

Hence, $f'(1)$ does not exist.

## General and Particular Solutions

Each individual solution of a differential equation is called a particular solution. A general solution of an ODE is a nonempty set of solutions specified by an expression which contains at least one parameter and which becomes a representation of a particular solution when its parameters are all replaced by numbers. The set of all solutions of a differential equation constitutes its complete solution.

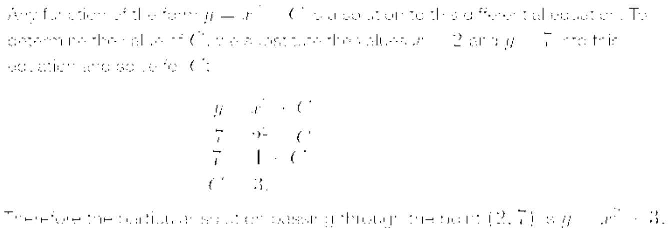
Families of Solution to the differential equation $y' = 2x$



Example:

Find the particular solution to the differential equation $y' = 2x$ passing through the point (2,7).

Solution:



Any function of the form $y = x^2 + C$ is a solution to the differential equation. To determine the value of $C$, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for $C$:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ 7 &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

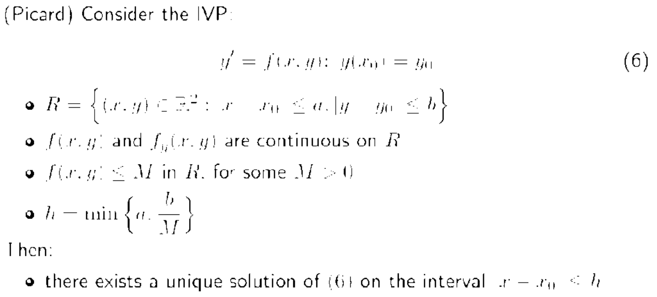
## Definition 3.1

The problem of determining solutions to ODE that satisfy some conditions is called an initial-vale problem (IVP). For first order equations, we wish to determine the conditions for the IVP

$$y' = f(x, y); \ y(x_0) = y_0 \tag{5}$$

to have a solution

## Theorem 3.2



(Picard) Consider the IVP:

$$y' = f(x, y); \ y(x_0) = y_0 \tag{6}$$

- $R = \left\{(x, y) \in \mathbb{R}^2 : |x - x_0| \le a, |y - y_0| \le b\right\}$
- $f(x, y)$ and $f_y(x, y)$ are continuous on $R$
- $|f(x, y)| \le M$ in $R$, for some $M > 0$
- $h = \min\left\{a, \frac{b}{M}\right\}$

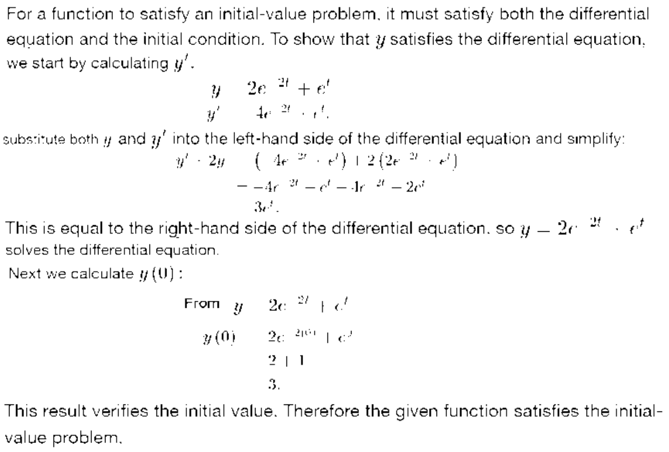
Then:

- there exists a unique solution of (6) on the interval $|x - x_0| \le h$

Example:

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem $y' + 2y = 3e^t, y(0) = 3$

Solution:



For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that $y$ satisfies the differential equation, we start by calculating $y'$.

$$\begin{aligned} y &= 2e^{-2t} + e^t \\ y' &= -4e^{-2t} + e^t. \end{aligned}$$

substitute both $y$ and $y'$ into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' + 2y &= (-4e^{-2t} + e^t) + 2(2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t + 4e^{-2t} + 2e^t \\ &= 3e^t. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation.

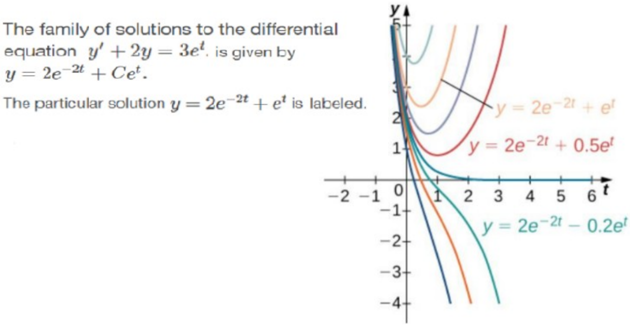
Next we calculate $y(0)$:

$$\begin{aligned} \text{From } y &= 2e^{-2t} + e^t \\ y(0) &= 2e^{-2(0)} + e^0 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Note: The initial-value problem consisted of two parts. The first part was the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C.

Illustration:



The family of solutions to the differential equation $y' + 2y = 3e^t$, is given by $y = 2e^{-2t} + Ce^t$.

The particular solution $y = 2e^{-2t} + e^t$ is labeled.